

# Mathematical Doodling: Reciprocal Fibonacci Numbers

Successive values in the infinite series of reciprocal Fibonacci numbers rapidly diminish, becoming arbitrarily small in the limit:

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In[76]:= Table[ $\frac{1}{\text{Fibonacci}[N]}$ , {N,1,20}]
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Out[76]= {1, 1,  $\frac{1}{2}$ ,  $\frac{1}{3}$ ,  $\frac{1}{5}$ ,  $\frac{1}{8}$ ,  $\frac{1}{13}$ ,  $\frac{1}{21}$ ,  $\frac{1}{34}$ ,  $\frac{1}{55}$ ,  $\frac{1}{89}$ ,  
 $\frac{1}{144}$ ,  $\frac{1}{233}$ ,  $\frac{1}{377}$ ,  $\frac{1}{610}$ ,  $\frac{1}{987}$ ,  $\frac{1}{1597}$ ,  $\frac{1}{2584}$ ,  $\frac{1}{4181}$ ,  $\frac{1}{6765}$ }
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In[77]:= N[%]
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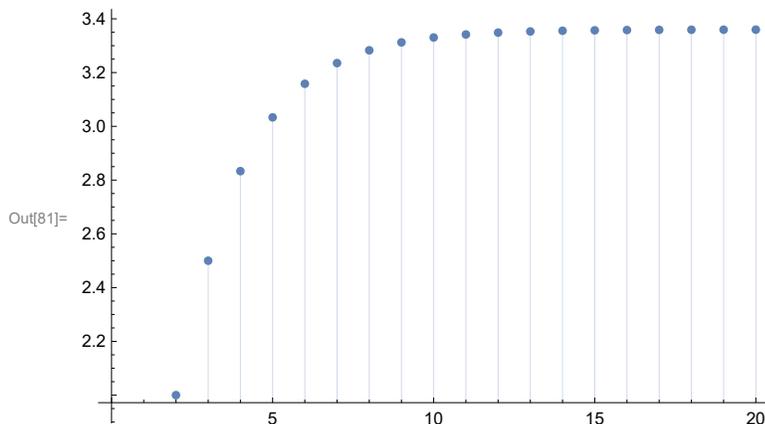
```
Out[77]= {1., 1., 0.5, 0.333333, 0.2, 0.125, 0.0769231, 0.047619, 0.0294118,  
0.0181818, 0.011236, 0.00694444, 0.00429185, 0.00265252, 0.00163934,  
0.00101317, 0.000626174, 0.000386997, 0.000239177, 0.00014782}
```

So the sum of the series appears likely to converge:

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In[54]:= Table[ $\sum_{n=1}^N \frac{1}{\text{Fibonacci}[n]}$ , {N,1,20}]//N
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Out[54]= {1., 2., 2.5, 2.83333, 3.03333, 3.15833, 3.23526, 3.28288, 3.31229, 3.33047, 3.3417,  
3.34865, 3.35294, 3.35559, 3.35723, 3.35825, 3.35887, 3.35926, 3.3595, 3.35965}
```

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In[81]:= ListPlot[Table[ $\sum_{n=1}^N \frac{1}{\text{Fibonacci}[n]}$ , {N,1,20}], Filling -> Axis]
```



## Sum of the Reciprocal Fibonacci Series

The direct attempt to find the limit of sum of series doesn't succeed, so we break the problem down into two parts; first, the series of even-indexed Fibonacci numbers:

$$\text{In[64]:= } \sum_{n=1}^N \frac{1}{\text{Fibonacci}[2n]}$$

$$\begin{aligned} \text{Out[64]= } & -\frac{\sqrt{5} \text{QPolyGamma}\left[\theta, 1, \frac{1}{4} (1 + \sqrt{5})^2\right]}{2 \text{Log}\left[\frac{1}{2} \times (3 + \sqrt{5})\right]} + \frac{\sqrt{5} \text{QPolyGamma}\left[\theta, 1 + N, \frac{1}{4} (1 + \sqrt{5})^2\right]}{2 \text{Log}\left[\frac{1}{2} \times (3 + \sqrt{5})\right]} + \\ & \frac{\sqrt{5} \text{QPolyGamma}\left[\theta, 1 - \frac{i \pi}{\text{Log}\left[\frac{1}{4} (1 + \sqrt{5})^2\right]}, \frac{1}{4} (1 + \sqrt{5})^2\right]}{2 \text{Log}\left[\frac{1}{2} \times (3 + \sqrt{5})\right]} - \\ & \frac{\sqrt{5} \text{QPolyGamma}\left[\theta, 1 + N - \frac{i \pi}{\text{Log}\left[\frac{1}{4} (1 + \sqrt{5})^2\right]}, \frac{1}{4} (1 + \sqrt{5})^2\right]}{2 \text{Log}\left[\frac{1}{2} \times (3 + \sqrt{5})\right]} \end{aligned}$$

In[65]= **FullSimplify[%56]**

$$\begin{aligned} \text{Out[65]= } & \frac{1}{\text{Log}[4] - 2 \text{Log}[3 + \sqrt{5}]} \\ & \sqrt{5} \left( \text{QPolyGamma}\left[\theta, 1, \frac{1}{2} \times (3 + \sqrt{5})\right] - \text{QPolyGamma}\left[\theta, 1 + N, \frac{1}{2} \times (3 + \sqrt{5})\right] - \text{QPolyGamma}\left[\theta, \right. \right. \\ & \left. \left. 1 + \frac{i \pi}{\text{Log}\left[\frac{2}{3 + \sqrt{5}}\right]}, \frac{1}{2} \times (3 + \sqrt{5})\right] + \text{QPolyGamma}\left[\theta, 1 + N + \frac{i \pi}{\text{Log}\left[\frac{2}{3 + \sqrt{5}}\right]}, \frac{1}{2} \times (3 + \sqrt{5})\right] \right) \end{aligned}$$

## Limit of the Even-Indexed Series

$$\text{In[68]:= } \text{Limit}\left[\sum_{n=1}^N \frac{1}{\text{Fibonacci}[2n]}, N \rightarrow \infty\right]$$

$$\text{Out[68]= } \left( \sqrt{5} \left( i \pi \text{Log}\left[\frac{1}{2} \times (3 + \sqrt{5})\right] + \text{Log}\left[\frac{1}{2} \times (3 - \sqrt{5})\right] \left( \text{QPolyGamma}\left[\theta, 1, \frac{1}{2} \times (3 + \sqrt{5})\right] - \right. \right. \right. \\ \left. \left. \left. \text{QPolyGamma}\left[\theta, 1 + \frac{i \pi}{\text{Log}\left[\frac{1}{2} \times (3 - \sqrt{5})\right]}, \frac{1}{2} \times (3 + \sqrt{5})\right]\right) \right) \right) / \\ \left( \text{Log}\left[\frac{1}{2} \times (3 - \sqrt{5})\right] \left( \text{Log}[4] - 2 \text{Log}[3 + \sqrt{5}] \right) \right)$$

$$\text{In[69]:= } \text{FibSumEven} = \text{N}[\text{Re}[\%], 10]$$

$$\text{Out[69]= } 1.535370509$$

## Limit of the Odd-Indexed Series

And now the sum of the odd-indexed series:

$$\text{In[82]:= } \text{Limit}\left[\sum_{n=0}^N \frac{1}{\text{Fibonacci}[2n+1]}, N \rightarrow \infty\right]$$

$$\text{Out[82]= } \left( (5 + \sqrt{5}) \left( \pi - \text{Im}\left[\text{QPolyGamma}\left[\theta, \frac{-i \pi + \text{Log}\left[\frac{2}{3 + \sqrt{5}}\right]}{\text{Log}[4] - 2 \text{Log}[3 + \sqrt{5}]}, \frac{1}{2} \times (3 + \sqrt{5})\right]\right] \right) + \right. \\ \left. \text{Im}\left[\text{QPolyGamma}\left[\theta, \frac{i \pi + \text{Log}\left[\frac{2}{3 + \sqrt{5}}\right]}{\text{Log}[4] - 2 \text{Log}[3 + \sqrt{5}]}, \frac{1}{2} \times (3 + \sqrt{5})\right]\right] \right) / \\ \left( 2 \sqrt{2 \times (3 + \sqrt{5})} \text{Log}\left[\frac{2}{3 + \sqrt{5}}\right] \right)$$

$$\text{In[83]:= } \text{FibSumOdd} = \text{N}[\text{Re}[\%], 10]$$

$$\text{Out[83]= } 1.824515157$$

## Sum of the Infinite Series of Reciprocal Fibonacci Numbers

So the sum of the infinite series of Fibonacci reciprocals is just the sum of the even-indexed and odd-indexed series:

$$\text{In[84]:= } \text{N}[\text{FibSumEven} + \text{FibSumOdd}, 10]$$

$$\text{Out[84]= } 3.359885666$$

## The Reciprocal Fibonacci Constant

It turns out that I am not the first to look into this question, which was addressed over 30 years ago by Horadam and others. The limit of the even-indexed series was given by Horadam 1988, Knopp 1990, Paszkowski 1997 and Finch 2003. Likewise the limit of the odd-indexed series was given in Horadam 1988, Griffin 1992, Zhao 1999, and Finch 2003.

The series sum even has a name: the Reciprocal Fibonacci Constant.

A related question raised by Paul Erdos is whether the constant is irrational; and it was proved to be so in 1989 by André-Jeannin

Looking at the above formulae, we find in several places the term  $\frac{1}{2} \times (3 + \sqrt{5})$ .

This is equivalent to  $1 + \phi$ , where:

In[94]=  $\phi = \text{Limit}[\text{Fibonacci}[N]/\text{Fibonacci}[N-1], N \rightarrow \infty]$

Out[94]=  $\frac{1}{2} \times (1 + \sqrt{5})$

is the

Out[93]= golden ratio

## References

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