

The SABR Stochastic Volatility Model

The SABR (Stochastic Alpha, Beta, Rho) model is a stochastic volatility model, which attempts to capture the volatility smile in derivatives markets. The name stands for "Stochastic Alpha, Beta, Rho", referring to the parameters of the model. The model was developed by Patrick Hagan, Deep Kumar, Andrew Lesniewski, and Diana Woodward.

The SABR Model Equations

The SABR model describes a single forward FF, such as a LIBOR forward rate, a forward swap rate, or a forward stock price. This is in contrast to the Black model which assumes that the forward is a lognormal random variable. It is convenient to express the SABR model in terms of the "forward variance" $v = \sigma^2$

, rather than the "forward volatility" σ . The SABR model is described by the following stochastic differential equations :

$$dF = \sigma F^{\beta} dW$$

$$d\sigma = \alpha \sigma dZ$$

$$dW dZ = \rho dt$$

where :

- FF is the forward price of the underlying asset.
- σ is the instantaneous volatility.
- α , β , and ρ are parameters to be fitted.
- W and Z are two correlated Wiener processes with correlation coefficient ρ .

The parameters β and α are constant and subject to the conditions $0 \leq \beta \leq 1$ and $\alpha \geq 0$. The parameter α represents volatility - like behavior for volatility. The correlation ρ represents the instantaneous relationship between the underlying asset and its volatility. The initial volatility $\sigma_{0 \times \square}$ determines the level of the ATM (at - the - money) implied volatility. Both ρ and β influence the slope of the implied skew, while the volatility of volatility α controls the curvature. This stochastic dynamics can be seen as a version of the CEV model with the skewness parameter β , which reduces to the CEV model when $\alpha = 0$. The parameter α is often referred to as "volvol," representing the lognormal volatility of the volatility parameter σ .

Relationship to Other Models

The SABR model is an extension of the Black model, which is itself an extension of the Black - Scholes model to handle interest rate derivatives. The Black - Scholes model assumes that volatil-

ity is constant, which is often not the case in the real world. The Black model extends the Black - Scholes model by allowing for a random forward price with constant volatility. The SABR model extends the Black model by allowing for a random forward price with stochastic volatility.

Applications of the Model

The SABR model is widely used by practitioners in the financial industry, especially in the interest rate derivatives market. It is used to calculate implied volatilities, to quote prices, and to risk manage and hedge new and existing positions. The popularity of the SABR model is mainly due to its empirical success, but also because it is extraordinary easy to calibrate.

Strengths and Limitations of the Model

The SABR model has the advantage of being able to reproduce the observed market phenomenon of the volatility smile, which is a pattern in which at - the - money options tend to have lower implied volatilities than in - or out - of - the - money options. The model is also relatively easy to calibrate to market data. However, the model also has several limitations. It assumes that the forward price and the volatility are lognormally distributed, which is not always the case in reality. It also assumes that the volatility is constant across strikes and maturities, which is not always the case in reality.

References

- Hagan, P., Kumar, D., Lesniewski, A., and Woodward, D. (2002). "Managing Smile Risk". *Wilmott Magazine*.
- Hagan, P., Lesniewski, A., and Woodward, D. (2010). "Probability Distribution in the SABR Model of Stochastic Volatility".
- B. Dupire, "Pricing with a Smile," *Risk*, 7, 1994 pp. 18–20.
- J. Oblój, "Fine-Tune Your Smile: Correction to Hagan et al.," *Wilmott Magazine*, May 2008.

Implementation in Wolfram Language

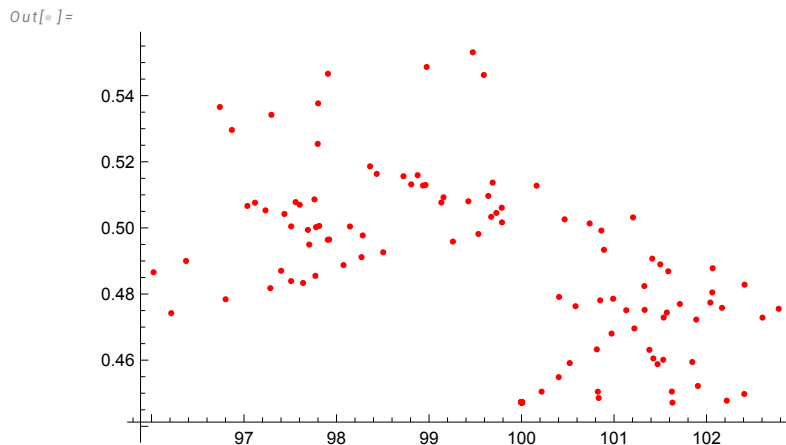
The SABR model can be implemented in the Wolfram Language as follows :

```
In[*]:= sabrModel[alpha_, beta_, rho_, F0_, v0_, T_, W_, Z_] :=
Module[{F, v, dW, dZ, dt = T / Length[W], results}, F = F0;
v = v0;
results = Table[{F, Sqrt[v]}, {Length[W]}];
Do[dW = Sqrt[dt] * W[[i]];
dZ = Sqrt[dt] * Z[[i]];
F = F + Sqrt[v] * F^beta * dW;
v = v + alpha * Sqrt[v] * dZ;
AppendTo[results, {F, Sqrt[v]}], {i, Length[W]}];
results]
```

This function takes as input the parameters of the SABR model (α, β, ρ), the initial forward price and volatility (F_0, v_0), the time horizon (T), and two sequences of random numbers (W, Z) represent-

ing the Wiener processes. It returns a sequence of pairs (F, σ) , representing the forward price and volatility at each time step. The function can be tested by generating some random numbers for W and Z :

```
In[ ]:= SeedRandom[1];
W = RandomReal[NormalDistribution[0, 1], 100];
Z = RandomReal[NormalDistribution[0, 1], 100];
ListPlot[sabrModel[0.1, 0.5, 0.25, 100, 0.2, 1, W, Z], PlotStyle -> Red]
```



Implied Volatility in the SABR Model

From Wikipedia:

We consider a [European option](#) (say, a call) on the forward F struck at K , which expires T years from now. The value of this option is equal to the suitably discounted expected value of the payoff $\max(F_T - K, 0)$ under the probability distribution of the process F_t .

Except for the special cases of $\beta = 0$ and $\beta = 1$, no closed form expression for this probability distribution is known. The general case can be solved approximately by means of an [asymptotic expansion](#) in the parameter $\varepsilon = T\alpha^2$. Under typical market conditions, this parameter is small and the approximate solution is actually quite accurate. Also significantly, this solution has a rather simple functional form, is very easy to implement in computer code, and lends itself well to risk management of large portfolios of options in real time.

It is convenient to express the solution in terms of the [implied volatility](#) σ_{impl} of the option. Namely, we force the SABR model price of the option into the form of the [Black model](#) valuation formula. Then the implied volatility, which is the value of the lognormal volatility parameter in Black's model that forces it to match the SABR price, is approximately given by:

$$\sigma_{\text{impl}} = \alpha \frac{\log(F_0/K)}{D(\zeta)} \left\{ 1 + \left[\frac{2\gamma_2 - \gamma_1^2 + 1/(F_{\text{mid}})^2}{24} \left(\frac{\sigma_0 C(F_{\text{mid}})}{\alpha} \right)^2 + \frac{\rho\gamma_1}{4} \frac{\sigma_0 C(F_{\text{mid}})}{\alpha} + \frac{2 - 3\rho^2}{24} \right] \varepsilon \right\},$$

where, for clarity, we have set $C(F) = F^\beta$. The value F_{mid} denotes a conveniently chosen midpoint between F_0 and K (such as the geometric average $\sqrt{F_0 K}$ or the arithmetic average $(F_0 + K)/2$). We have also set

$$\zeta = \frac{\alpha}{\sigma_0} \int_K^{F_0} \frac{dx}{C(x)} = \frac{\alpha}{\sigma_0(1-\beta)} (F_0^{1-\beta} - K^{1-\beta}),$$

and

$$\gamma_1 = \frac{C'(F_{\text{mid}})}{C(F_{\text{mid}})} = \frac{\beta}{F_{\text{mid}}},$$

$$\gamma_2 = \frac{C''(F_{\text{mid}})}{C(F_{\text{mid}})} = -\frac{\beta(1-\beta)}{(F_{\text{mid}})^2},$$

The function $D(\zeta)$ entering the formula above is given by

$$D(\zeta) = \log \left(\frac{\sqrt{1 - 2\rho\zeta + \zeta^2} + \zeta - \rho}{1 - \rho} \right).$$

Alternatively, one can express the SABR price in terms of the [Bachelier's](#) model. Then the implied normal volatility can be asymptotically computed by means of the following expression:

$$\sigma_{\text{impl}}^n = \alpha \frac{F_0 - K}{D(\zeta)} \left\{ 1 + \left[\frac{2\gamma_2 - \gamma_1^2}{24} \left(\frac{\sigma_0 C(F_{\text{mid}})}{\alpha} \right)^2 + \frac{\rho\gamma_1}{4} \frac{\sigma_0 C(F_{\text{mid}})}{\alpha} + \frac{2 - 3\rho^2}{24} \right] \varepsilon \right\}.$$

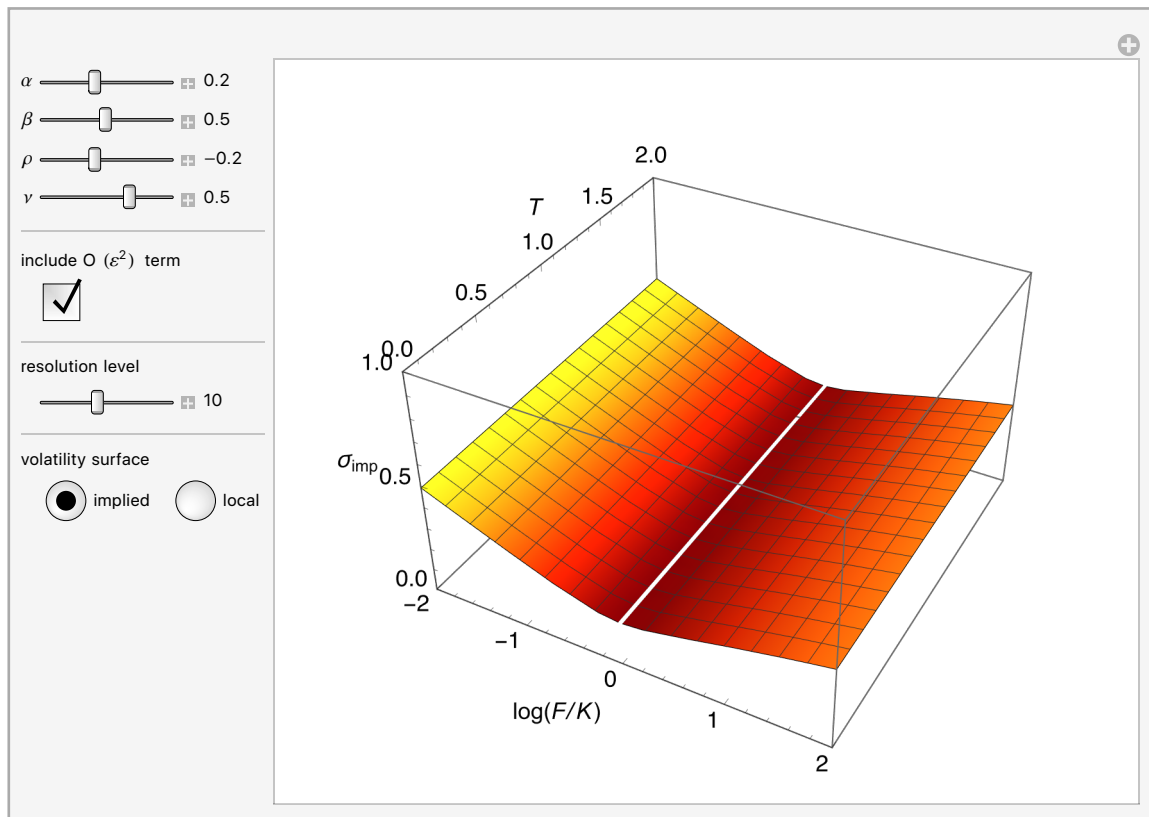
It is worth noting that the normal SABR implied volatility is generally somewhat more accurate than the lognormal implied volatility.

The approximation accuracy and the degree of arbitrage can be further improved if the equivalent volatility under the [CEV model](#) with the same β is used for pricing options.^[2]

Stochastic and Local Volatility Dynamics

Volatility Dynamics

Out[*]=



Thomas Mazzoni

"Implied and Local Volatility Dynamics in the SABR Model"

<http://demonstrations.wolfram.com/ImpliedAndLocalVolatilityDynamicsInTheSABRModel/WolframDemonstrationsProject>

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```
In[*]:= ManToGif[man_, name_String, step_Integer] := Export[name <> ".gif",
  Import[Export[name <> Which[$OperatingSystem == "MacOSX", ".mov",
    $OperatingSystem == "Windows", ".avi"], man], "ImageList"][[1 ;; -1 ;; step]]
```

```
In[*]:= ManToGif[man, "D:\\Dropbox\\Mathematica\\Finance\\SABR", 4]
```

Out[*]=

D:\Dropbox\Mathematica\Finance\SABR.gif